Identificar funções diferenciáveis

Teorema da função composta:

Se
$$\mathbb{R}^n \xrightarrow[\text{dif. em } a]{g} \mathbb{R}^p \xrightarrow[\text{dif. em } g(a)]{f} \mathbb{R}^m$$
$$f\circ g$$

$$a \longmapsto g(a) \longmapsto f(g(a))$$

então, $f\circ g$ é dif. em a

e $$\boxed{D(f\circ g)(a) = Df(g(a))\,Dg(a)}$$

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Exemplo:

$$f(x,y) = h\big(\underbrace{x+y,\; x+2y,\; 3x+y}_{g(x,y)}\big)$$

$h : \mathbb{R}^3 \longrightarrow \mathbb{R}$, dif.

$g : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$g(x,y) = (x+y,\ x+2y,\ 3x+y)$$

$$f(x,y) = h(g(x,y))$$

$$\begin{tikzcd}
(x,y) \arrow[r, mapsto, "g"] \arrow[rr, bend right, "f = h \circ g"'] & (x+y,\ x+2y,\ 3x+y) \arrow[r, mapsto, "h"] & h(x+y, x+2y, 3x+y)
\end{tikzcd}$$

$$\underset{1\times 2}{Df(x,y)} = \underset{1\times \underline{3}}{Dh(g(x,y))}\ \underset{\underline{3}\times 2}{Dg(x,y)}$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}_{(x,y)} = \begin{bmatrix} \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

$$Df(1,0) = Dh\big(g(1,0)\big)\,Dg(1,0)$$

$$g(1,0) = (1,\ 1,\ 3)$$

$$Df(1,0) = \underbrace{Dh(1,1,3)}_{\text{tem de ser dada}}\ \underbrace{Dg(1,0)}_{\text{a calcular}}$$

h não é conhecida!

g é conhecida!

$\boxed{\text{Dado!!!}}$ $Dh(1,1,3) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

$$Df(1,0) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$g(x,y) = (x+y,\ x+2y,\ 3x+y)$$

$$= \begin{bmatrix} 12 & 8 \end{bmatrix}$$

$$f(x,y) = h(x+y,\ x+2y,\ 3x+y)$$

$$f(x,y) = h\big(\underbrace{u(x,y), v(x,y), w(x,y)}_{g(x,y)}\big)$$

$f: \mathbb{R}^2 \to \mathbb{R}$

$h: \mathbb{R}^3 \to \mathbb{R}$ $\qquad h(u,v,w)$

$u, v, w : \mathbb{R}^2 \to \mathbb{R}$

$$(x,y) \longmapsto (u(x,y), v(x,y), w(x,y)) \longrightarrow h(\dots)$$

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$Df(x,y) = \begin{bmatrix} \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{bmatrix}_{(u,v,w)} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix}_{(x,y)}$$

$$\| $$

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}_{(x,y)}$$

$$\frac{\partial f}{\partial x} = \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial h}{\partial w}\frac{\partial w}{\partial x}$$

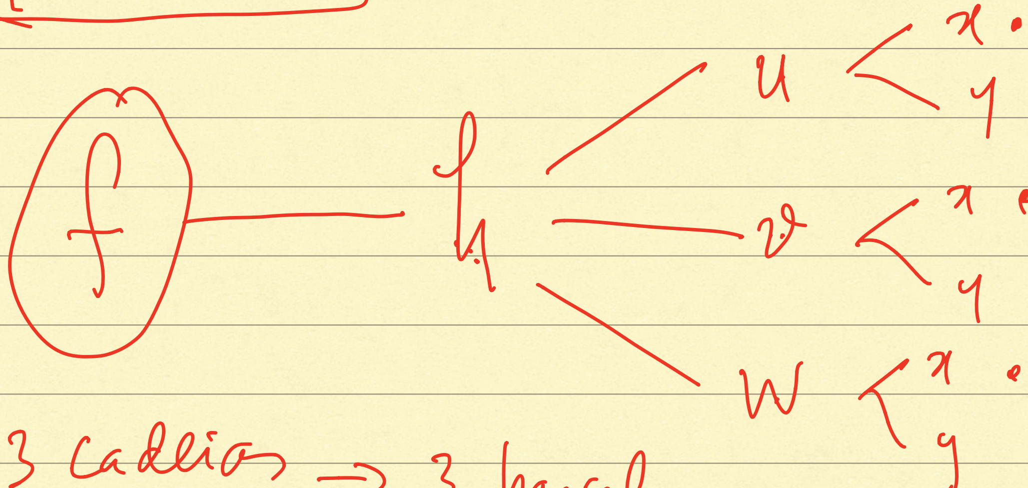
$$\frac{\partial f}{\partial y} = \frac{\partial h}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial h}{\partial w}\frac{\partial w}{\partial y}$$

$$f(x,y) = h(u(x,y), v(x,y), w(x,y))$$

Regra de Cadeia

$h\,u\,x$, $h\,v\,x$, $h\,w\,x$

Cadeias de símbolos!



3 cadeias → 3 parcelas

$$f(x,y) = h\Big(\overbrace{x+y}^{u(x,y)},\ \overbrace{x+2y}^{v(x,y)},\ \overbrace{3x+y}^{w(x,y)}\Big)$$

$$\frac{\partial f}{\partial x} = \frac{\partial h}{\partial u}\,1 + \frac{\partial h}{\partial v}\,1 + \frac{\partial h}{\partial w}\,3$$

$$= \overset{=1}{\frac{\partial h}{\partial u}} + \overset{=2}{\frac{\partial h}{\partial v}} + 3\,\overset{=3}{\frac{\partial h}{\partial w}} = 12$$

(x,y) $\qquad$ (u,v,w)

$\boxed{(1,0)}$ $\qquad$ $\boxed{(1,1,3)}$

Atenção aos pontos!

Exemplo:

$$f(x,y) = e^{x^2+y^2+xy} = h(u(x,y))$$

$$h(u) = e^u \qquad f = h \circ u$$

$$u(x,y) = x^2+y^2+xy$$

$$Df(x,y) = Dh(u(x,y))\,Du(x,y)$$

$$f(x,y) = h(u(x,y))$$

$$\frac{\partial f}{\partial x} = \frac{\partial h}{\partial u}\,\frac{\partial u}{\partial x}$$

$h(u) = e^{u}$

$u(x,y) = x^2+y^2+xy$

$$\frac{\partial f}{\partial x} = e^{u(x,y)}(2x+y)$$

$$= e^{x^2+y^2+xy}(2x+y)$$

$$\frac{\partial f}{\partial x} = (2x+y)\,e^{x^2+y^2+xy}$$

$$\begin{tikzcd}
(x,y) \arrow[r, mapsto] \arrow[rr, bend right, "f(x,y)"'] & u(x,y) \arrow[r, mapsto] & h(u(x,y))
\end{tikzcd}$$

Teorema da função composta: Demonstração:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "g", "\text{dif. em } a"'] & \mathbb{R}^p \arrow[r, "f", "\text{dif. em } g(a)"'] & \mathbb{R}^m \\
a \arrow[r, mapsto] \arrow[d] & g(a) \arrow[r] \arrow[d] & f(g(a)) = (f\circ g)(a) \\
a+h \arrow[r] & g(a+h) = g(a)+k & \text{(ver livro)}
\end{tikzcd}$$

$$\boxed{g(a+h) - g(a) = k}$$

$$f(g(a)+k) - f(g(a)) - Df(g(a))k = o_f(k)$$

$$k = g(a+h) - g(a) = Dg(a)h + o_g(h)$$

etc.

$$f(g(a)+k) = f(g(a+h))$$
$$= (f \circ g)(a+h)$$
$$f(g(a)) = (f \circ g)(a)$$

$$(f \circ g)(a+h) - (f \circ g)(a) - Df(g(a))k = o_f(k)$$

$$k = g(a+h) - g(a) = Dg(a)h + o_g(h)$$

etc...

$$\boxed{D(f \circ g)(a) = Df(g(a))\,Dg(a)}$$